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Optimal design of passive linear suspension using genetic algorithm

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Abstract

In this paper the genetic algorithm (GA) method is applied to the optimization problem of a linear one-degree-of-freedom (1-DOF) vibration isolator mount and the method is extended to the optimization of a linear quarter car suspension model. A novel criterion for selecting optimal suspension parameters is presented. An optimal relationship between the root mean square (RMS) of the absolute acceleration and the RMS of the relative displacement is found. Although the systems are linear, it is difficult to find such optimal relation analytically. The optimum solution is obtained numerically by utilizing GA and employing a cost function that seeks minimizing absolute acceleration RMS sensitivity to changes in relative displacement RMS. The combination of RMS and absolute acceleration sensitivity minimization produces optimal suspension that is robust to broadband frequency excitation. The GA method increases the probability of finding the global optimum solution and avoids convergence to a local minimum which is a drawback of gradient-based methods. Given allowable mount relative displacement (working space), designers can use the results to specify the optimal mount and suspension. The cost function employed can be extended to optimize multi-DOF (MDOF) and non-linear vibrating mechanical systems in frequency domain. Applying the method to a linear quarter car model illustrates the applicability of the method to MDOF systems. An example is given to demonstrate the optimality of the solution obtained by the GA technique.

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1. Introduction

Isolation systems design presents a challenge to engineers because of the conflicting criteria involved in their design. For example, in the automotive industry it is desired to reduce engine vibration and ultimately the dynamic forces transmitted from the engine to the frame. Usually this is achieved by using engine mounts or vibration isolators that connect the engine to the chassis. On one hand, reducing vibration transmissibility entails using mounts as soft as possible. On the other hand, constraints on engine deflection due to physical limitations restrict mount stiffness to a lower boundary. Consequently, the design of passive mounts and isolation systems can be treated as an optimization problem [1].

In this paper, an optimization criterion for the design of vibration isolation systems is introduced. One of the advantages of this optimization criterion is that the optimal values of the dynamical parameters of the system do not approach trivial solutions. Genetic algorithm (GA) is used to numerically implement the method. GA is a stochastic optimization technique based on the mechanics of natural evolution and survival of the fittest strategy found in biological organisms. Two examples illustrate the effectiveness of the technique. First a non-dimensionalized model of a linear mount is optimized. Next, the developed theory is extended and applied to optimize a quarter car model as an example of a multi-degrees-of-freedom (MDOF) system.

Figs. 1 and 2 depicts a linear vibration isolation system and a linear quarter car model, respectively. The goal is to develop a design chart utilizing GA to evaluate the optimum values of

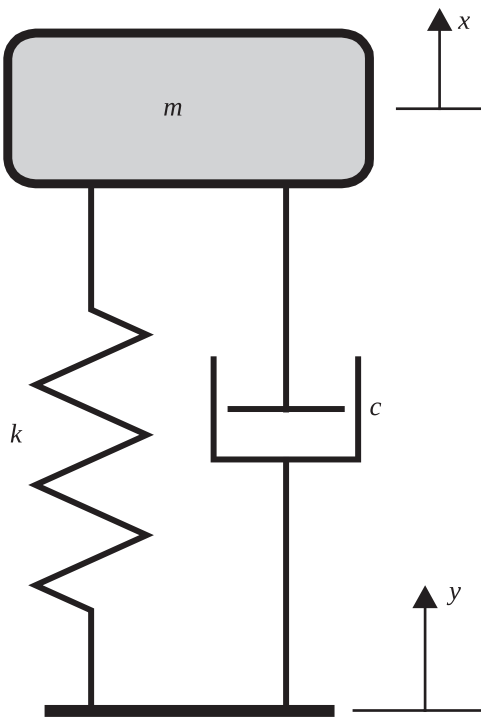

Fig. 1. Mathematical model of a linear mount.

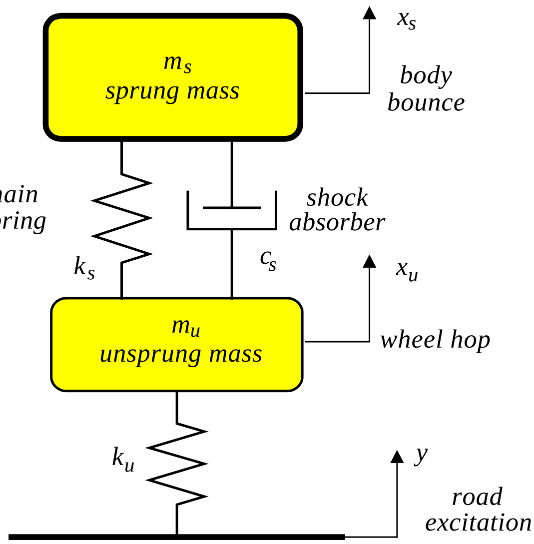

Fig. 2. Schematic of a quarter car model.

the main suspension stiffness and damping parameters for maximum isolation of the upper mass from a harmonic base excitation in the frequency domain. To this end, a cost function using RMS of both absolute acceleration and relative displacement is defined. Then the cost function is minimized to create a design chart enabling the selection of the optimum natural frequency and damping ratio.

1.1. Genetic algorithm

GA is a well-known method for global optimization of complex systems. The start of GA can be traced back to 1950s, but the work done in 1970s by John Holland at the University of Michigan led to GA, as we know it today [2,3]. In simple terms the algorithm represents a search strategy based on the mechanics of natural selection and reproduction in biological systems. The search procedure is derived from the process of natural selection and evaluation originally observed and documented by Charles Darwin. The philosophy of "survival of the fittest" has been adopted, implemented numerically, and developed for the general problem of optimization in which natural evolution and adaptation to environment variation is simulated mathematically. Because of the inherent advantage of being able to proceed with a large population of designs, the method facilitates arrival at the globally optimal solution [4].

GA starts with a set of randomly selected potential solutions to the problem at hand, and makes them evolve by iteratively applying a set of stochastic operators, known as selection, crossover and mutation. The technique relies on objective function (fitness) evaluation [5]. The better solution has higher fitness value. No gradient information is required; only evaluation of the objective function and the constraints are necessary to determine fitness. Such a derivative freeness technique makes GA versatile and gives it the ability to deal with problems with a complicated objective function where derivative is difficult to obtain or unattainable (non-differentiable function). The stochastic and randomness nature of GA avoids the gradient-based optimization methods drawback of getting trapped in local optima.

Only a brief description of the theory of GA is given in this paper as it is by now a well-established optimization technique. The interested reader may consult Refs. [2,6,7] for many practical and theoretical aspects of GA. DeJong [6] studied the use of GA in general function optimization. He showed that the ability of the GA to learn from the history and exploit the environment provided the basis of its effectiveness in optimization. Recent years have witnessed an exponential growth in the use of GA in a vast variety of sciences and engineering fields. Forrest [8] collected a good summary of GA applications to science and engineering problems up to 1993. In vibration isolation systems the works reported by East et al. [9], Baumal et al. [10], and Baldanzini et al. [11] may be mentioned.

1.2. Vibration isolation

The most important function of an isolator is to reduce the magnitude of motion transmitted from a vibrating foundation to the equipment, or to reduce the magnitude of force transmitted from the equipment to its foundation, both in time and frequency domain [12]. Different methods exist to address time and frequency domain isolation system optimization [13]. Time domain optimization deals with the dynamic response and transient characteristics of the system. However, frequency domain optimization is concerned with the steady state performance of the system. Optimization in the frequency domain is essential, specially when the excitation has a different combination of frequencies. Optimization of the structure from a harmonic excitation may be used for any type of periodic excitation, considering that any periodic excitation can be expanded as a Fourier series of harmonic excitations.

Optimization in the frequency domain involves analysis of the frequency response function (FRF) of the system, which relates the steady state response to the disturbance input. Derivation of the required FRF is sometimes cumbersome particularly for statistical characteristics, and is near impossible for non-linear system. In this case the, optimization process relies significantly on numerical simulation rather than on an analytical solution.

Optimization of vibration isolation systems has been the subject of a vast amount of research [14,15]. In the first few decades of the last century, researchers established the theory of vibration isolation. Den Hartog [16] pointed out that prior to the middle of 1930s, vibration isolation theory had not yet been introduced into the curriculum of technical schools. Now there are many theories, and various passive, semiactive and active vibration isolation systems are available.

In the simplest approach to the problem of estimating the effectiveness of a vibration mount (see Fig. 1), researchers assume that the engine or equipment to be isolated is a rigid mass m and the mount is a massless mechanically paralleled spring and damper of stiffness k and resistance c. The parameters m, k, and c are considered constant and independent of the frequency. For foundation-excited vibration, we assume that the engine does not affect the vibratory foundation velocity, whether the engine is rigidly or resiliently attached. This assumption is equivalent to the assumption of an infinitely stiff and massive foundation [17].

The equations that govern the linear model of an isolator with a harmonic base excitation, and the relevant transfer functions of the linear model may be found in any mechanical vibration texts such as Den Hartog's [16]. The non-dimensionalized equation of motion for the system is

$$\ddot{x}_r + 2\xi\omega_n\dot{x}_r + \omega_n^2 x_r = Y\omega^2 \sin(\omega t), \tag{1}$$

where the parameters of Eq. (1) are related to the physical parameters of the system by

$$\xi = \frac{c}{c_c}, \quad \omega_n = \sqrt{\frac{k}{m}} = 2\pi f_n, \quad c_c = 2m\omega_n, \quad x_r = x - y. \tag{2}$$

The most important transfer functions for the system are: absolute displacement, γ, relative displacement, λ, and absolute acceleration, a, which are defined as follows:

$$\gamma = \left|\frac{X}{Y}\right| = \frac{\sqrt{1 + (2\xi(\omega/\omega_n))^2}}{\sqrt{(1 - (\omega/\omega_n)^2)^2 + (2\xi(\omega/\omega_n))^2}}, \tag{3}$$

$$\lambda = \left|\frac{X - Y}{Y}\right| = \frac{(\omega/\omega_n)^2}{\sqrt{(1 - (\omega/\omega_n)^2)^2 + (2\xi(\omega/\omega_n))^2}}, \tag{4}$$

$$a = \left|\frac{\ddot{X}}{Y}\right| = \frac{\omega^2\sqrt{1 + (2\xi(\omega/\omega_n))^2}}{\sqrt{(1 - (\omega/\omega_n)^2)^2 + (2\xi(\omega/\omega_n))^2}}. \tag{5}$$

Figs. 3 and 4 depict the variation of the system relative displacement transmissibility, and absolute acceleration amplitude transmissibility versus frequency ratio.

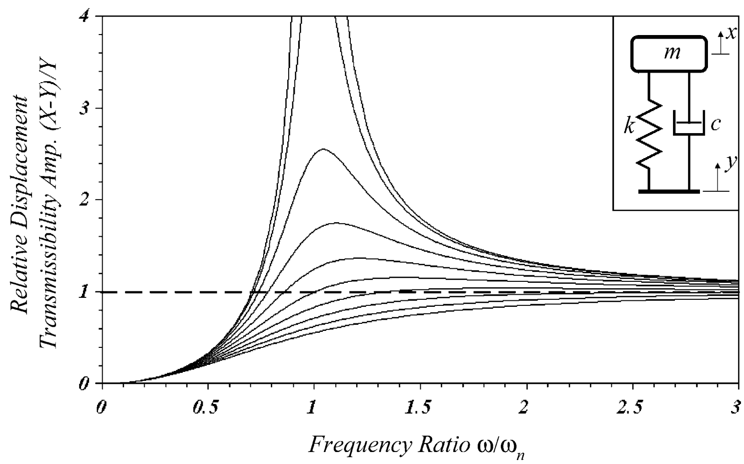

Fig. 3. Frequency response of the relative displacement of linear 1-DOF base excited system.

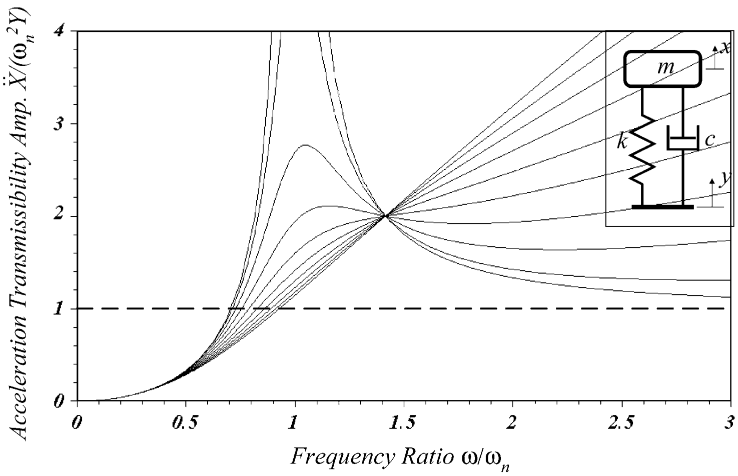

Fig. 4. Frequency response of the absolute acceleration of linear 1-DOF base excited system.

2. Optimization problem

There are numerous practical applications where it is desired to isolate vibrating machines and devices from the surrounding structure by minimize the vibration transmitted away from the vibratory system. For example, engine mounts are typically used to isolate the engine from the frame structure on which it is mounted. The goal of optimization processes is to achieve the best possible vibration attenuation under various conditions. The procedure involves three major tasks; first to choose what measure should be minimized to best depict the problem under study. The next question of interest is to decide which parameters are allowed to vary during the optimization. Finally, one has to decide what constraints must be satisfied in order to avoid trivial solutions to the problem.

The choice of the objective function is paramount, since it determines which isolation system design is best or optimum. There are various approaches to the selection of the objective function, but none is universally accepted yet. Most of the criteria used in objective function for optimization of vibratory systems are based on the acceleration, jerk, and displacement. The reduction of the absolute acceleration is important in the optimization of suspensions since it measures the transmitted force to the sprung mass. Hence, absolute acceleration is an essential parameter of every cost function in the vibration isolation optimization theory. Relative displacement transmissibility is another significant quantity to be taken into consideration. It measures the ratio of the relative deflection amplitude of the isolator to the excited displacement amplitude imposed at the foundation. A vibration isolator produces a reduction in absolute acceleration, and absolute displacement vibrations by permitting deflection of the isolator [17]. The relative deflection is a measure of the clearance (known as working space, travel space, or rattle-space) required in the isolator. The clearance should be bounded due to the physical consideration in the mechanical design.

For mechanical systems, the frequency domain of interest is usually between zero and 20 Hz. Selected stiffness and damping should be optimum over the entire frequency domain. Developing a passive vibration isolator requires a frequency-averaged optimum design. Hence, the optimum stiffness and damping may be found by using some type of averaging characteristic in the frequency domain. The root mean square (RMS) can be used as the average over the frequency domain [0,20] Hz. The choice of constraints restricts the possibilities of candidate designs. However, the choice of objective function, and constraints are limited by the practical consideration.

For the system shown in Fig. 1, it is generally desired to select ξ and ω_n such that the absolute acceleration (or relative displacement) of the system is minimized and the relative displacement (or the absolute acceleration) does not exceed a prescribed level. The RMS of the acceleration and the RMS of the relative displacement are defined by the functions

$$R = \sqrt{\frac{1}{40\pi}\int_0^{40\pi} a^2\,\mathrm{d}\omega}, \quad \eta = \sqrt{\frac{1}{40\pi}\int_0^{40\pi} \lambda^2\,\mathrm{d}\omega}. \tag{6, 7}$$

There is a tradeoff between the acceleration and relative motion, which is exploited to achieve the optimal isolator. Fig. 5 illustrates this tradeoff. The ratio of RMS of the absolute acceleration to the RMS of the relative displacement is a monotonically increasing function of ω_n and ξ. If the relative displacement RMS is kept constant, then the acceleration RMS increases with an increase in of ω_n or ξ. Also if the acceleration RMS is kept constant then the relative displacement RMS decreases with an increase in ω_n or ξ. Hence absolute acceleration and relative displacement cannot both be minimized at the same time. In other words, decreasing absolute acceleration necessarily increases the relative displacement and vice versa. Therefore, in the absence of a constraint, the optimum design is the trivial solution of $\omega_n = 0$ and $\xi = 0$ (the no connection case).

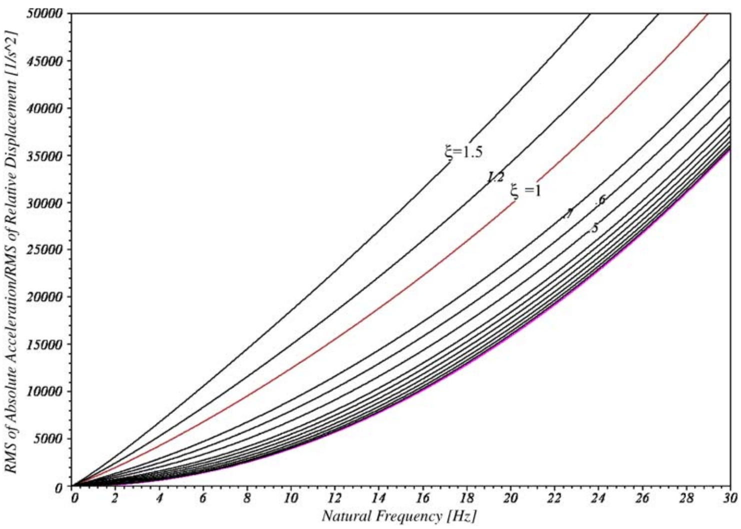

Fig. 5. Tradeoff between the RMS of the absolute acceleration and the RMS of the relative displacement.

By considering these design specifications, a general design optimization statement for the mount system can now be given. An appropriate optimization criterion maybe defined as

Optimizatin criterion: minimize RMS of absolute acceleration with respect to RMS of relative displacement for a given value of RMS of relative displacement.

The result of this optimization criterion is an optimal curve in the R–η plane. Select a desired value for relative displacement as the allowable mount deflection and find the associated values for ξ and ω_n.

3. Genetic algorithm optimization procedure

In this section a description of the implementation of GA to the optimization of a linear mount system is given. GA is a subset of evolutionary algorithms that model and mimic biological processes to find optimal solutions of highly complex problems. GA draws analogy to the natural process of reproduction, natural selection and evolution in biological population, where genetic characteristics stored in chromosomal strings evolves over generations to give individuals a better chance of survivability in a static or changing environment. This chromosomal configuration represents the generational memory and is partially transferred and altered when members of the population reproduced. The basic idea of a GA is simple. First, a population of individuals is created in a computer (typically stored as binary strings in the computer's memory), and then the population is evolved with use of the principles of variation, selection, and inheritance [8].

The three basic processes that affect the chromosomal makeup in natural evolution are crossover of genetic information between the reproduction parents, an occasional mutation of genetic information, and survival of the fittest to reproduce in upcoming generations. Crossover process exchanges genetic structure between the parents and allows for beneficial genes to be represented in the offspring. Mutation is a sudden and infrequent alternation in chromosomal makeup, which causes new traits to surface in individuals. Individuals with favorable qualities have better chance to adapt and survive, and therefore procreate.

GA, in a fashion analogous to their natural counterpart, uses chromosome-type representations of feasible solutions of the problem to explore the searching space for improved solutions. GA incorporates a bias reproduction strategy, where members of the population that are deemed most fit are preferred for reproduction and given higher opportunity to strengthen the chromosomal composition of the offspring generation. This approach is implemented by assigning a fitness value or scale indicating the goodness of an individual of the population in a given generation during the evolution process. The objective function serves as excellent candidate in measuring individual's fitness.

Major component of GAs including encoding scheme, fitness evaluation, parents selection, crossover operators and mutation operators are briefly explained next in the contest of the mount optimization.

3.1. Principle of encoding scheme

The first step is to transform points in the parameter space into bit string representations. By converting each parameter into its binary equivalent, it may be mapped into a fixed-length string of 0's and 1's (*gene*). Clearly, the string (*gene*) length determines the numerical precision with which this parameter may be represented. For the mount problem, each variable x_i is represented by a 16-digit binary number with maximum and minimum values of the design variable mapped to the maximum and minimum of the binary number as follows:

$$x_{i\min} = 0000000000000000,$$

$$x_{i\max} = 1111111111111111.$$

Two such binary numbers are needed for the mount optimization problem to represent damping ratio and natural frequency $(x_1, x_2) = (\xi, \omega_n)$ with a domain of possible values limited to $0 < \xi < 1.5$ and $0 < \omega_n < 100$ rad/s. The two genes are then placed end to end to create a 32-digit string concatenated binary string *chromosome* of 0's and 1's

$$chromosome = \left[\underbrace{1010111\cdots}_{gene_1=\xi} \underbrace{0011101\cdots}_{gene_2=\omega_n} \right].$$

This 32-digit string represents one design of the mount, so there are 2^{16} possibilities for each variable and 2^{32} possible design. A sequence of such strings can be introduced to construct a population of designs. An initial population of 100 designs was randomly generated.

3.2. Cost function and fitness evaluation

GA use selection, crossover, and mutation operators to breed good solutions. "Goodness" of the solution is measured by so-called "fitness function." The fitness function is based on the objective function of the problem and must be non-negative [18]. The fitness should be evaluated for each design in every generation and it depends on the objective function value. The objective function in the mount case should reflect how close is the design to the condition given in the optimization criterion. If η_g is the given value of RMS of relative displacement where it is needed to set the system such that

$$\left.\frac{\partial R}{\partial \eta}\right|_{\eta_g} = 0, \quad \left.\frac{\partial^2 R}{\partial \eta^2}\right|_{\eta_g} > 0 \tag{8}$$

and (ξ_j, ω_{n_j}) is the jth individual at any generation. Based on (ξ_j, ω_{n_j}) two more points (ξ_i, ω_{n_i}) and (ξ_k, ω_{n_k}) are defined, where

$$\xi_i = \xi_j - e, \quad \omega_{n_i} = \omega_{n_j}, \quad \xi_k = \xi_j + e, \quad \omega_{n_k} = \omega_{n_j}, \quad e \ll 1. \tag{9}$$

The equation of a parabola passing through points (ξ_i, ω_{n_i}), (ξ_j, ω_{n_j}), and (ξ_k, ω_{n_k}) is

$$R = A\eta^2 + B\eta + C, \tag{10}$$

where

$$
\begin{aligned}
A &= -\frac{R_i(\eta_j-\eta_k)+R_j(\eta_k-\eta_i)+R_k(\eta_i-\eta_j)}{-\eta_i^2(\eta_j-\eta_k)-\eta_j^2(\eta_k-\eta_i)-\eta_j^2(\eta_k-\eta_i)},\\
B &= -\frac{R_i(\eta_j^2-\eta_k^2)+R_j(\eta_k^2-\eta_i^2)+R_k(\eta_i^2-\eta_j^2)}{-\eta_i^2(\eta_j-\eta_k)-\eta_j^2(\eta_k-\eta_i)-\eta_j^2(\eta_k-\eta_i)},\\
C &= \frac{-R_i\eta_j\eta_k(\eta_j-\eta_k)+R_j\eta_k\eta_i(\eta_k-\eta_i)+R_k\eta_i\eta_j(\eta_i-\eta_j)}{-\eta_i^2(\eta_j-\eta_k)-\eta_j^2(\eta_k-\eta_i)-\eta_j^2(\eta_k-\eta_i)}.
\end{aligned}
\tag{11}
$$

The minimum of parabola (11) occurs at

$$\eta_m = -\frac{1}{2}\frac{R_i(\eta_j^2-\eta_k^2)+R_j(\eta_k^2-\eta_i^2)+R_k(\eta_i^2-\eta_j^2)}{R_i(\eta_j-\eta_k)+R_j(\eta_k-\eta_i)+R_k(\eta_i-\eta_j)}. \tag{12}$$

Now the cost function is defined such that η_j, η_m and η_g coincide with each other

$$J_j = (\eta_j-\eta_m)^2+(\eta_m-\eta_g)^2+(\eta_g-\eta_j)^2, \tag{13}$$

along with the conditions

$$R_j > R_i, \quad R_j < R_k. \tag{14}$$

3.3. Selection

After evaluating the fineness of each member of the current population, a selection process for individuals to participate in the creation of the next generation is in order. The selection should be biased toward individuals with higher fitness value analogous to survival of the fittest in natural selection. Selection for reproduction among members of higher fitness ensures moving the search toward producing more fit members in the population and eliminating the less fit ones. First the population is ranked according to their fitness. A mating pool consisting of 50% of the population with individual with the highest fitness is created. Members are selected from the mating pool and paired (i.e., $\{parent_1, parent_2\}$) with selection probability proportional to their fitness value. If J_r is the fitness measure of the rth member, it can be alloted a probability of $J_r/\sum_{j=1}^{n} J_j$, where n is the population size. Self-pairing is not permitted. The paired individuals are used to create new individuals through crossover operators to replace the discarded ones.

3.4. Crossover

Crossover is the exchange of design characteristics among randomly selected pairs from the parent pool. There are many types of crossover; the most general one (i.e., uniform crossover) is briefly introduced here and implemented for the mount optimization. Uniform crossover looks at each bit in the parents and randomly assigns the bit from one parent to one offspring and the bit from the other parent to the other offspring. First a random mask is generated. This mask is a random vector of ones and zeros and is the same length as the parents. When the bit in the mask is 0, then the corresponding bit in $parent_1$ is passed to $offspring_1$ and the corresponding bit in $parent_2$ is passed to $offspring_2$. When the bit in the mask is 1, then the corresponding bit in $parent_1$ is passed to $offspring_2$ and the corresponding bit in $parent_2$ is passed to $offspring_1$:

$Parent_1$	**01001101101011100001010101110111**,
$Parent_2$	11000110110110111111010011001001,
$Mask$	00011010001111001101101000101111,
$Offspring_1$	**010**00**1**1**110**0110**10**11**0**10**1**0**101**0**1**1001,
$Offspring_2$	110**01**1**0**011**1011**11**00**1**10**1**0**011**1**0**0111**.

3.5. Mutation

This step diversifies the population so that different areas of the parameters space can be explored and also prevents the solution from premature convergence. A mutation operator is capable of spontaneously generating new chromosomes. The most common way of implementing mutation is by switching a 0 with a 1 or vice versa with a probability equal to the very low given mutation rate. The mutation rate was taken to be 10% of the total number of binary digits in the whole population. The size of the population passed from one generation to the other remains constant. A complete iteration or new generation of designs is formed after completing all of the above steps.

4. Application of the genetic algorithm

GA is applied to obtain the optimal design charts for the linear engine mount by running the GA for a hundred points of given relative displacement RMS in the range $0.22 \leqslant \eta_g \leqslant 1.35$. For each point, the algorithm stops after a fixed number of iteration set to be 500 iterations. The result is the optimal values of the damping ratio and natural frequency corresponding to each given relative displacement RMS.

Fig. 6 depicts the connected optimal points in the $RMS(\ddot{x})$–$RMS(x_r)$ plane that satisfies the optimality condition (8). Connecting the optimal points together makes the optimal line. The optimal line starts from a point between 1.4 and 1.5 on $RMS(x_r)$ axis showing a soft mount (low stiffness and damping), and ends at a point close to ($7000\,\mathrm{s}^{-2}$) on $RMS(\ddot{x})$ axis showing a hard mount (rigid connection). Point $(1, 0)$ indicates a no connection condition between the mass and the base. The optimality condition (8) does not converge to the point $(1, 0)$ that the other optimal conditions converge to.

Fig. 7 illustrates the variation of the natural frequency and damping ratio versus $RMS(x_r)$. After finding an optimal point in Fig. 6, the designer can find the dynamical parameters of the mount from Fig. 7. The graphical illustration of the relationship between the natural frequency and damping ratio on the optimal line is shown in Fig. 8. It demonstrates that at optimum conditions, increasing the damping ratio is followed by an increase in the natural frequency, and vise versa. The natural frequency reaches a saturating level. Increasing damping ratio beyond 0.6 does not affect the value of the optimal natural frequency. The level of acceleration for the optimum mount always lies below the level of acceleration for a hard mount, which is desirable. Its level of relative displacement may be less or greater than a soft mount. Reducing the working

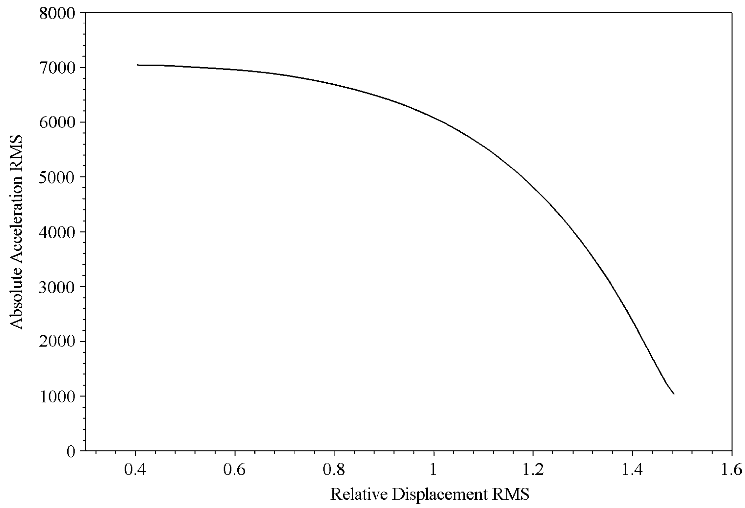

Fig. 6. Optimal curve in $RMS(\ddot{x})$–$RMS(x_r)$ plane.

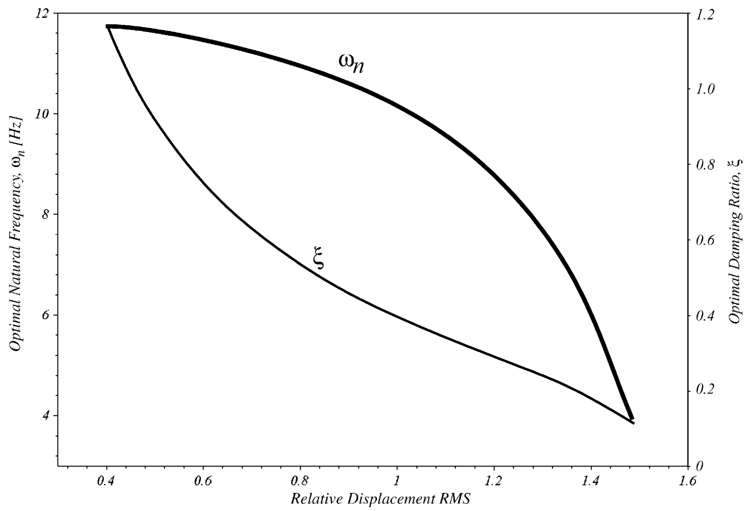

Fig. 7. Optimal curves in ω_n–$RMS(x_r)$ and ξ–$RMS(x_r)$ plane

space requires increasing the acceleration response, which can be done by using a stiffer mount and higher damping ratio.

Although, the level of R changes slowly for $0<\eta<1$, it changes at a higher rate for $\eta>1$. In other words, at high natural frequencies, the optimum RMS acceleration becomes insensitive to damping. If the limit value of the RMS of the relative displacement (or acceleration) is known, then the intersection of the corresponding vertical (horizontal) line in Fig. 6 with the line of optimum indicates the optimum value of ξ and ω_n and the corresponding level of acceleration (relative displacement).

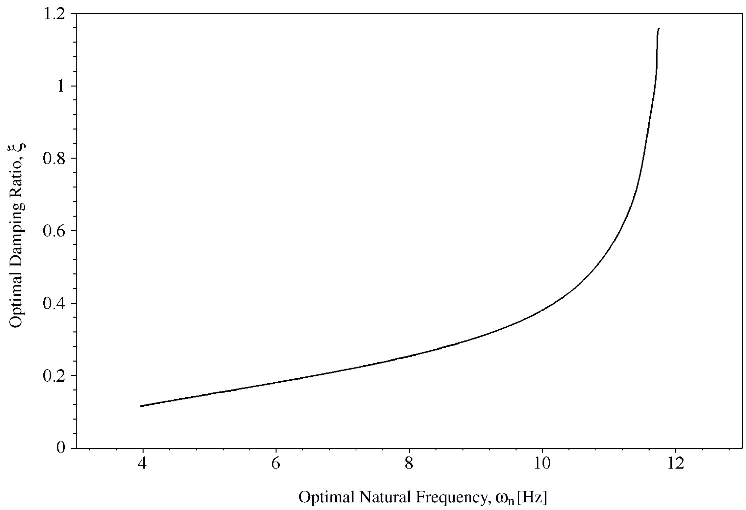

Fig. 8. Optimal curve in ω_n–ξ plane.

Note that the RMS of the absolute acceleration R and the RMS of the relative displacement η are functions of two variables ω_n and ξ as indicated in Eqs. (4)–(7):

$$R = f_1(\omega_n, \xi), \quad \eta = f_2(\omega_n, \xi). \tag{15, 16}$$

Hence, a pair of (ω_n, ξ), uniquely determines R and η. Theoretically, the variables can be changed to define any of these four characters R, η, ξ, and ω_n as a function of the other two variables such as

$$R = g_1(\eta, \xi), \quad R = g_2(\omega_n, \eta). \tag{17, 18}$$

Consequently, ξ and ω_n can be regarded as a surface in the spaces of (R, ω_n, η), and (R, η, ξ). The functions f_1 and f_2 or g_1 and g_2 determine the dynamical behavior of the system. Figs. 9 and 10 depict the surfaces f_1 and f_2 using Eqs. (15) and (16). Figs. 11 and 12 also illustrate the behavior of the absolute acceleration RMS, R, using the surfaces g_1 and g_2 specified in Eqs. (17) and (18).

Each pair of (ω_n, ξ), (ω_n, η), or (η, ξ) indicates a characteristic point on surfaces (15), (17), or (18) respectively. The characteristic point uniquely determines the RMS of the frequency response of the system. The point cannot leave the surfaces, but can slide on the surfaces. It means that the upper and lower half-spaces, above and below the characteristic surfaces, are meaningless and there are no real behaviors corresponding to those half-spaces. Hence, only two of the four characters, R, η, ξ, and ω_n are independent and it is possible to reduce the number of independent characters to one by introducing an optimal relationship between two of them. The design procedure reduces to calculate the other three variables when the fourth one (usually the relative displacement RMS, η) is given.

Although there is no extremum on the functions f_1, f_2, g_1, and g_2, (see Figs. 9–12), it is possible to find a curve on the surface g_2, passing through the minimum of intersection of g_2, and the

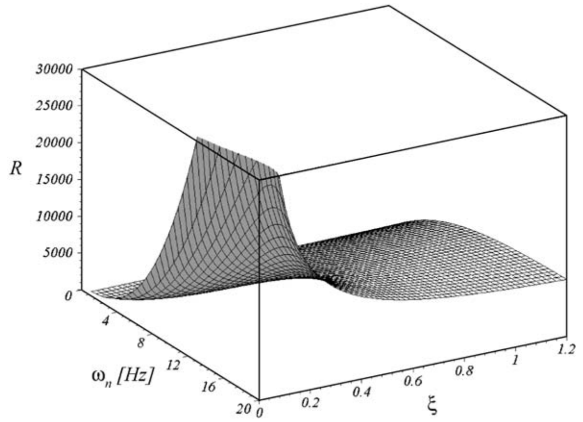

Fig. 9. Illustration of f_1, the RMS of the absolute acceleration, R, for a linear mount, as a function of natural frequency, ω_n, and damping ratio, ξ.

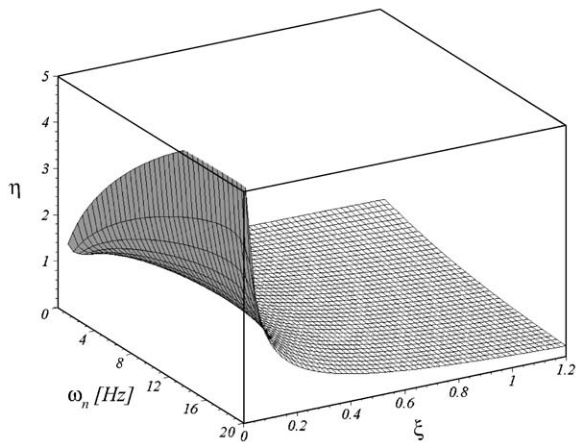

Fig. 10. Illustration of f_2, the RMS of the relative displacement, η, for a linear mount, as a function of natural frequency, ω_n, and damping ratio, ξ.

planes indicated by ω_n. This space curve is defined as the optimal curve. The optimal curve shown in Fig. 13 depicts a relationship between ω_n and ξ that makes R minimum with respect to η, when ω_n is given. In other words, for any specific value of ω_n there is a solution for $\partial g_2/\partial\eta = 0$ such that $\partial^2 g_2/\partial\eta^2 > 0$. The shape of the optimal curve could be seen in the (R, η)-plane view of the surface g_2, as illustrated in Fig. 14.

If in Fig. 13, $\hat{\mathbf{e}}_\eta, \hat{\mathbf{e}}_{\omega_n}, \hat{\mathbf{e}}_R$ are the unit vectors along the axes η, ω_n, and R respectively, and ∇g_2 is the gradient of the surface g_2, then the optimal curve is defined by

$$\nabla g_2 \cdot \hat{\mathbf{e}}_\eta = 0, \tag{19}$$

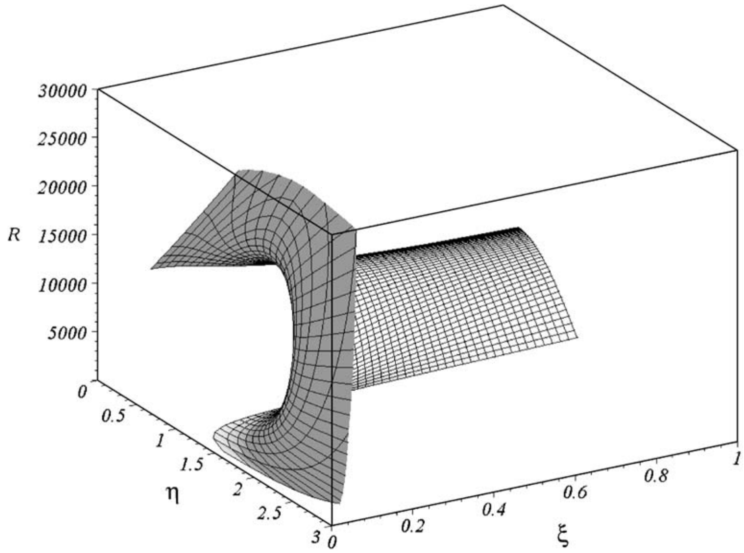

Fig. 11. Illustration of g_1, the RMS of the absolute acceleration, R, for a linear mount, as a function of relative displacement RMS, η, and damping ratio, ξ.

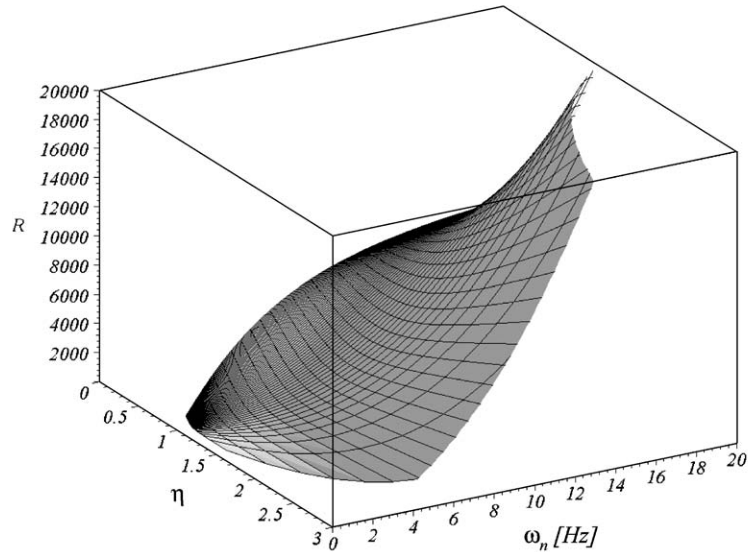

Fig. 12. Illustration of g_2, the RMS of the absolute acceleration, R, for a linear mount, as a function of relative displacement RMS, η, and natural frequency, ω_n.

which shows that the gradient of g_2 has no component on η axes. Fig. 15 illustrates how the optimal curve defined by Eq. (12) passes through the minimum points. On the optimal curve, R is just a variable of ω_n, and is not sensitive to small changes in η. Since η is a measure of working space, this property is more important for actual working conditions.

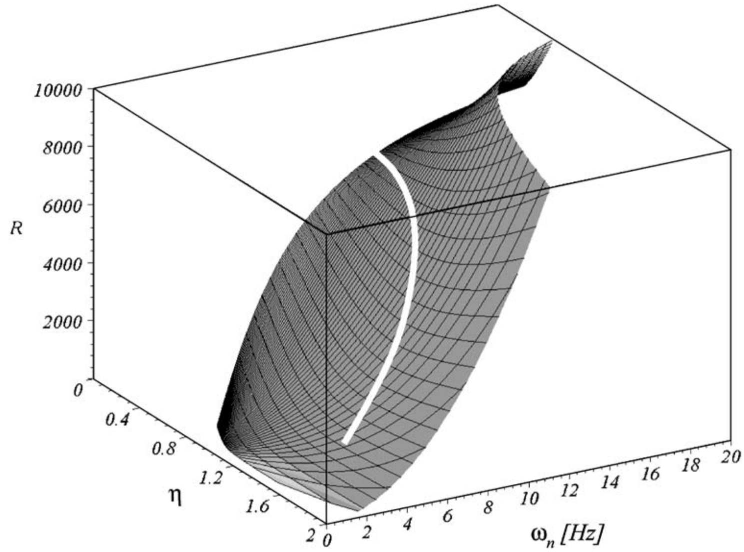

Fig. 13. Illustration of the optimal curve on the surface g_2, the acceleration RMS, R, for a linear mount, as a function of relative displacement RMS, η, and natural frequency, ω_n.

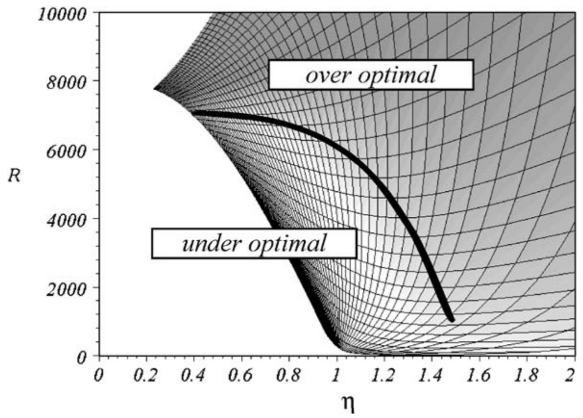

Fig. 14. Illustration of the (R, η)-plane view of the surface $R = g_2(\omega_n, \eta)$.

5. Quarter car optimization

Next a linear 2-DOF quarter car model is analyzed. The same optimization criterion is employed to obtain the optimal damping and stiffness values for the main suspension by minimizing the RMS of the absolute acceleration of the sprung mass with respect to the relative displacement RMS. The RMS values are used to create design curves for the suspension parameters, which are very useful particularly in the presence of physical constraints such as a limit on relative displacement.

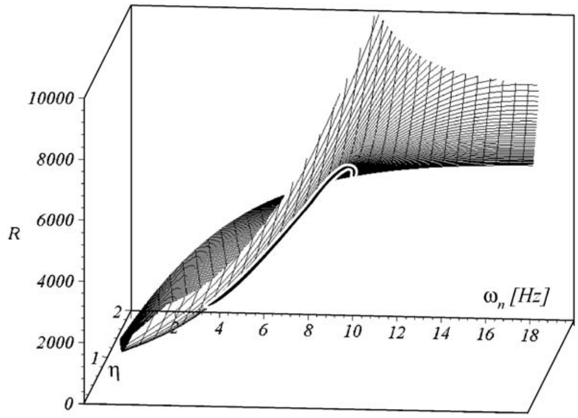

Fig. 15. Illustration of the optimal curve defined by $\nabla g_2 \cdot \hat{\mathbf{e}}_\eta = 0$ on the surface g_2, the acceleration RMS, R, for a linear mount, as a function of relative displacement RMS, η, and natural frequency, ω_n.

Schematically, a vehicle suspension mechanism can be represented using a linear system consisting of two solid mass m_s and m_u denoted as sprung and unsprung masses, linked to each other by a spring mechanism of stiffness k_s, and a shock absorber with viscous damping coefficient c_s. The solid mass m_u, represents the wheel; it is in direct contact with the ground through a spring of stiffness k_u [19]. This model is shown in Fig. 2. The damping requirement is in conflict with ride comfort; i.e., high and low damping must alternate within the ranges of excitation frequency in order to provide good vibration isolation over the entire frequency range.

The governing differential equations of motion for the model are

$$m_s\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_s\right) + cs\left(\frac{\mathrm{d}}{\mathrm{d}t}(x_s - x_u)\right) + k_s(x_s - x_u) = 0, \tag{20}$$

$$m_u\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_u\right) + c_s\left(\frac{\mathrm{d}}{\mathrm{d}t}(x_u - x_s)\right) + (k_s + k_u)x_u - k_s x_s = k_u y. \tag{21}$$

In order to investigate the frequency response, and develop an optimization procedure based on frequency response, harmonic excitation is assumed, $y = Y\mathrm{e}^{\mathrm{i}\omega t}$, and a periodic solution is sough after of the form $x_s = X_s\mathrm{e}^{\mathrm{i}\omega t}$, $x_u = X_u\mathrm{e}^{\mathrm{i}\omega t}$, where X_s, X_u are complex amplitudes.

The following dimensionless characteristics are introduced:

$$\omega_u = \sqrt{\frac{k_u}{m_u}}, \quad \omega_s = \sqrt{\frac{k_s}{m_s}}, \quad r = \frac{\omega}{\omega_s}, \quad \alpha = \frac{\omega_s}{\omega_u}, \quad \varepsilon = \frac{m_s}{m_u}, \quad \xi = \frac{c_s}{2\omega_s m_s} \tag{22}$$

and after some manipulations, the transmissibilities $\mu = X_s/Y$, $\tau = X_u/Y$ and $\eta = (X_s - X_u)/Y$, for sprung, unsprung and wheel travel, respectively, are found

$$\mu^2 = \frac{4\xi^2 r^2 + 1}{[r^2(r^2\alpha^2 - 1) + (1 - (1+\varepsilon)r^2\alpha^2)]^2 + 4\xi^2 r^2(1 - (1+\varepsilon)r^2\alpha^2)^2}, \tag{23}$$

$$\tau^2 = \frac{4\xi^2 r^2 + 1 + r^2(r^2 - 2)}{[r^2(r^2\alpha^2 - 1) + (1 - (1+\varepsilon)r^2\alpha^2)]^2 + 4\xi^2 r^2(1 - (1+\varepsilon)r^2\alpha^2)^2}, \tag{24}$$

$$\eta^2 = \frac{r^4}{[r^2(r^2\alpha^2 - 1) + (1 - (1+\varepsilon)r^2\alpha^2)]^2 + 4\xi^2 r^2(1 - (1+\varepsilon)r^2\alpha^2)^2}. \tag{25}$$

Eqs. (23)–(25) show that the transmisibilities μ, τ, and η are functions of four essential variables; mass ratio ε, damping ratio ξ, natural frequency ratio α, and excitation frequency ratio r. It is important to note that the frequency ratios α and r only appears in even powers.

For a quarter car model, it is known that if $m_u < m_s$, then $\varepsilon > 1$. Typical mass ratio for the commercial vehicles lies in the range 3–8 with small cars near the former figure and large cars near the large one [20]. Also note that the excitation frequency ω would be equal to ω_u, when $r = 1/\alpha$, and would be equal to ω_s, when $r = 1$. For a real model, the order of magnitude of the stiffness is $k_u > k_s$, also $\omega_u > \omega_s$, and then $\alpha < 1$. Therefore, $r > 1$ at $\omega = \omega_u$.

The following optimization criterion is defined, similar to the previous one used for the linear mount, to optimize the suspension of the linear quarter car model:

Optimization criterion: minimize RMS of absolute acceleration U with respect to RMS of relative displacement Φ, where $U = \mathrm{RMS}(u)$, $\Phi = \mathrm{RMS}(\eta)$, and u is the absolute acceleration of the sprung mass

$$u = \frac{\ddot{X}_s}{\omega_1^2 Y} = r^2\alpha^2\mu. \tag{26}$$

Using this optimality condition, an optimal relationship between absolute acceleration RMS and relative displacement RMS is found. The result is an optimal curve in the U–Φ plane. Select a desired value for relative displacement as the traveling space, and find the associated values for ξ and α at the intersection of the associated vertical line with the optimal curve. Mathematically, it is equivalent to the constrained minimization

$$\frac{\partial U}{\partial \Phi} = 0, \quad \frac{\partial^2 U}{\partial \Phi^2} > 0. \tag{27}$$

For a real problem, the values of mass ratio, ε, and tire frequency ω_u are fixed and the designers seek to find the optimum values of α and ξ. The parameters α and ξ include the unknown stiffness of the main spring and the unknown damping of the main shock absorber, respectively.

Applying GA to this problem produces the optimal relationship between α and ξ. Figs. 16–18 show the results graphically. Fig. 16 represent the optimal curve in the U–Φ plane. The associated value of α and ξ are depicted in Figs. 17(a) and (b), and the relationship between α and ξ is indicated in Fig. 18.

6. Optimization example

Verification of the result can be done by analyzing the frequency response behavior of the system using optimal parameters. The analysis of the frequency response is a good measure for comparing the suspension parameters in order to find the effect of the RMS optimized parameters on the steady state response. A harmonic base excitation is applied to the system. Note that the

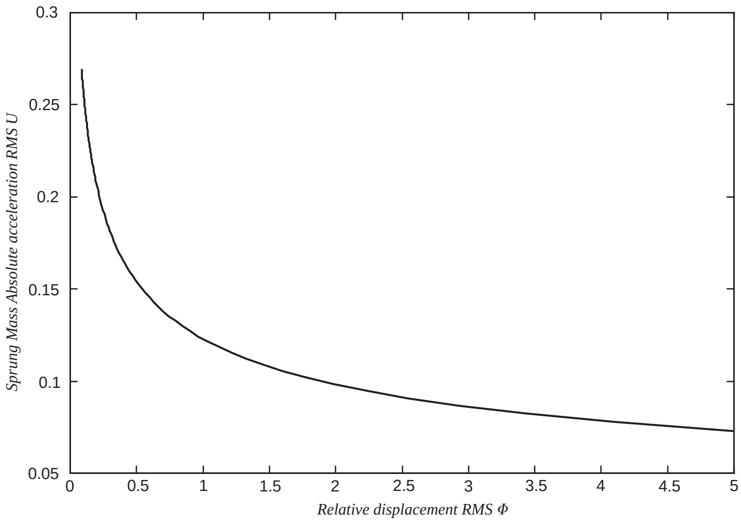

Fig. 16. Optimal curve in-plane U–Φ for a quarter car model.

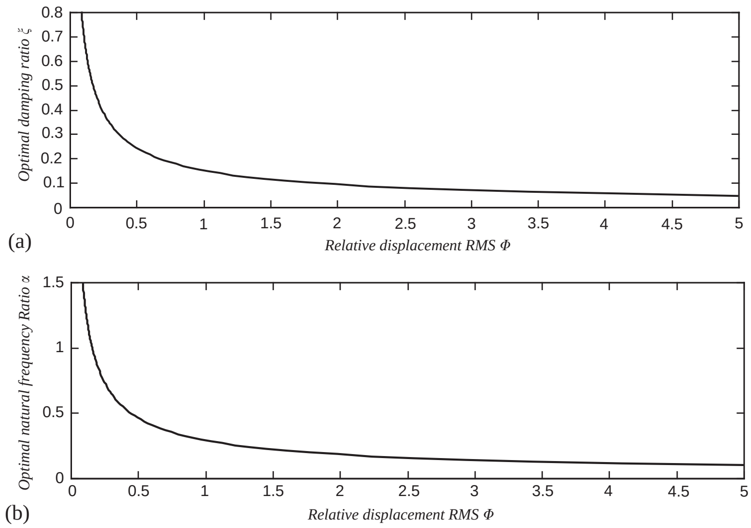

Fig. 17. (a) Optimal ξ versus Φ for a quarter car model and (b) optimal α versus Φ for a quarter car model.

result based on RMS optimization analysis is equivalent to variance optimization of a white noise random excitation with zero mean value. Therefore, suspensions on the line of minima have an optimal behavior in the random domain.

The behavior of the mount system for five different cases, indicated in Figs. 19 and 20, are analyzed. Point 1 is a random picked point on the optimal line. Points 2 and 3 are two alternative points with the same natural frequency as point 1. Points 4 and 5 are two points with the same damping ratio as point 1. The relative displacement RMS and absolute acceleration RMS for

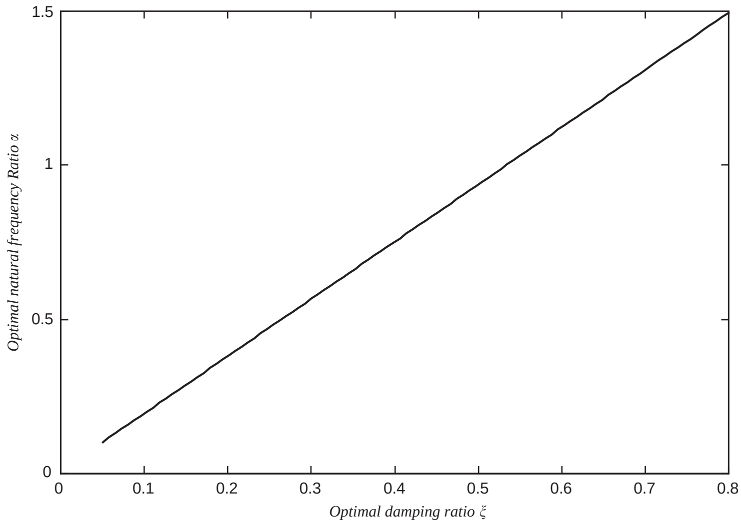

Fig. 18. Relationship between α and ξ at optimal condition for a quarter car model.

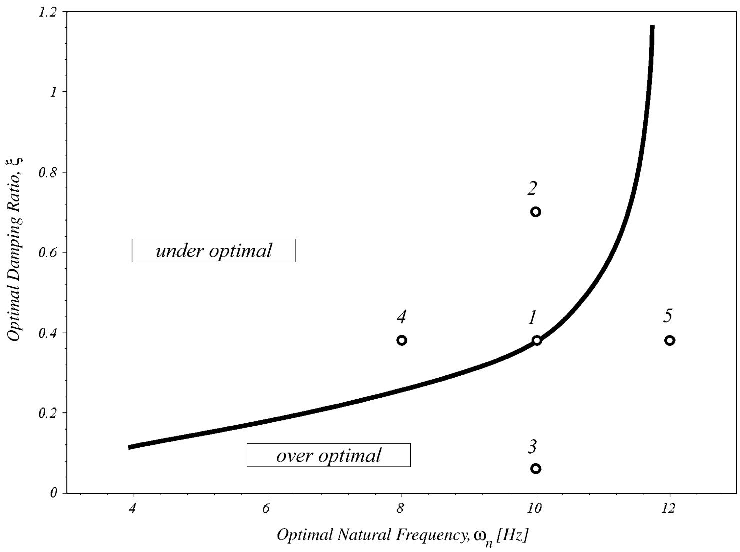

Fig. 19. Five mounts in ω_n–ξ plane.

points 1–5 are plotted in Fig. 20. Points 1, 2, 4, and 5 are close enough to compare, but point 3 is an odd point. Therefore, points 3 indicates a significant different suspension which is a low damped suspension. According to the optimal prediction, the behavior of point 1 is better than the other trial points. The dynamical parameters of the selected mount systems are presented in Table 1. Figs. 21 and 22 compare the absolute acceleration frequency response a, and relative displacement frequency response λ for the five point. Furthermore, Fig. 23 depicts the frequency response of the absolute displacement γ.

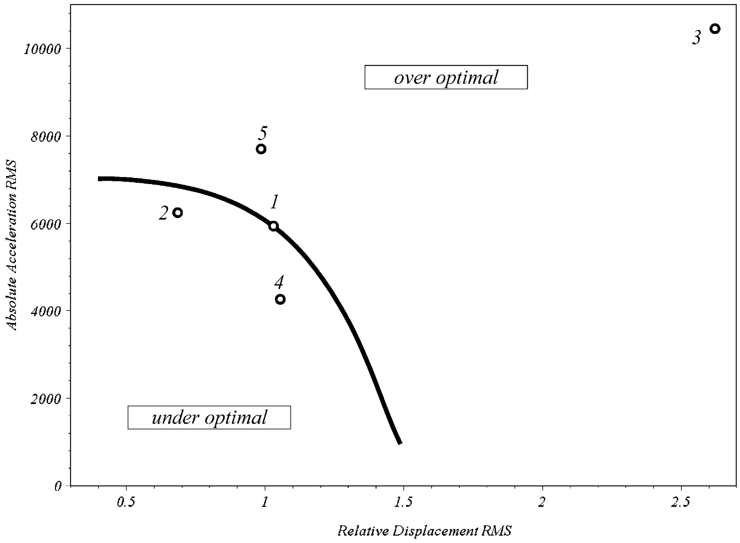

Fig. 20. Five mounts in $RMS(\ddot{x})$–$RMS(x_r)$ plane.

Table 1
Numerical values of dynamical parameters for five different suspensions shown in Figs. 19 and 20

	R	η	f_n	ξ
Point 1	5945	1.02967	10	0.38
Point 2	6252	0.6870	10	0.7
Point 3	10447	2.6225	10	0.06
Point 4	4267	1.0550	8	0.38
Point 5	7701	0.9853	12	0.38

As previously mentioned in Section 1, the frequency range $0 \leqslant \omega_n \leqslant 20\,\text{Hz}$ is of interest since mechanical vibrations are located in this frequency bandwidth. Fig. 21 shows that the mounts related to points 3 and 5 have a higher acceleration frequency response at the working frequency range which confirm the prediction of Fig. 20. This characteristic can be termed "overoptimal." The points below the optimal curve in the plane are termed "underoptimal," because they show less acceleration RMS compared to optimal points with the same relative displacement RMS. Figs. 22 and 23 illustrate that point 3 has an unacceptable peak value of displacement. Suspension 5 has a high displacement at high frequencies. Even though that the acceleration frequency response related to point 2 is less than the other points at low frequency, it has the steepest gradient and has a higher level of acceleration at high frequency. The same situation occurs in the relative displacement frequency response. This phenomenon comes from the high damping ratio in system number 2. In other words, point 2 shows a lazy system, and it is apparent in Fig. 24 when comparing the time response of the relative displacement to a unit step input.

Point 4 has a lower acceleration RMS than point 1, and it seems that it has a better behavior than the optimal point 1. To show that point 1 is optimal, we may disturb the trial points 1

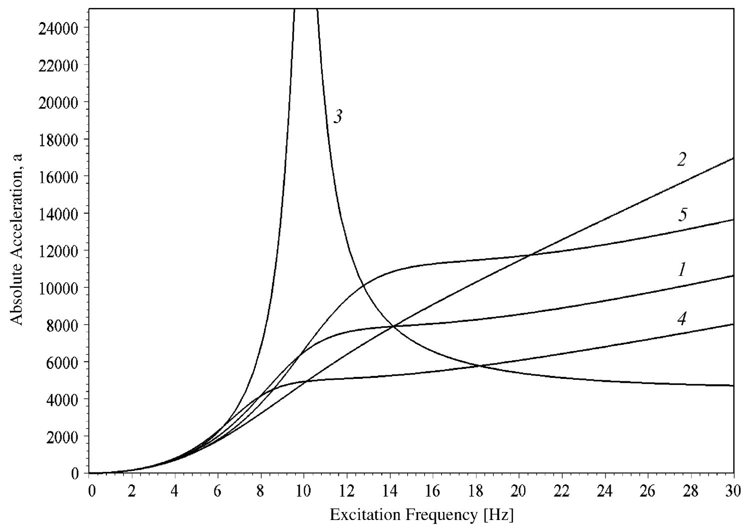

Fig. 21. Linear mount absolute acceleration frequency response a, comparison of five suspensions.

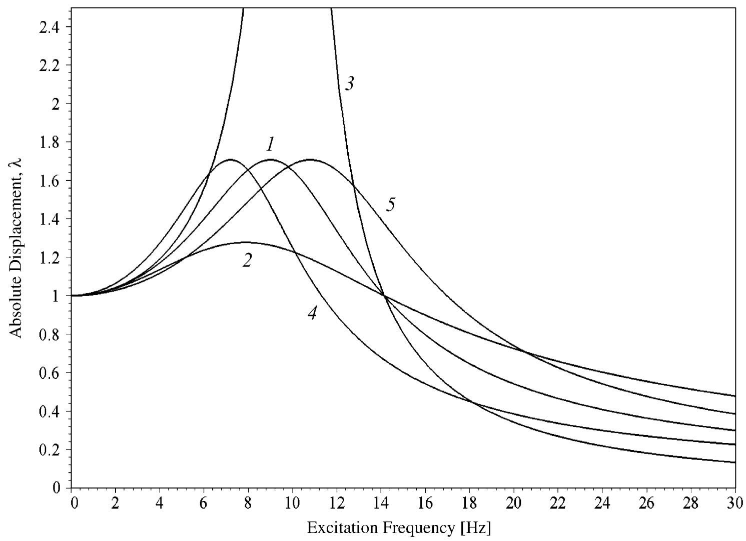

Fig. 22. Linear mount relative displacement frequency response λ, comparison of five suspensions.

through 5 by changing their position in Fig. 20, horizontally, and compare the behavior of new trial points $1'$ through $5'$ (see Fig. 25). Point 1 is more resistant to change in mount characteristics. The value of ξ and ω_n for new trial points are indicated in Table 2.

The acceleration frequency response of the system of perturbed and unperturbed trial points are shown in Fig. 26. Point 1 has the least sensitivity to a change of relative displacement constrain. This perturbation is typical in the real world due to aging and changing of working conditions.

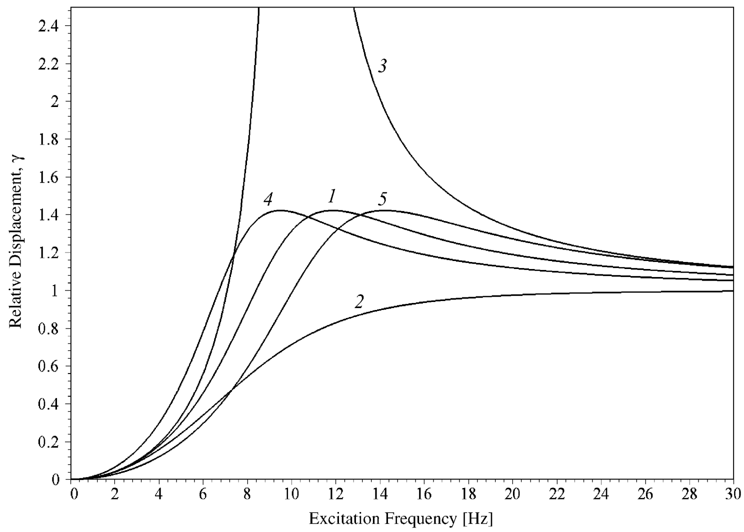

Fig. 23. Linear mount absolute displacement frequency response γ, comparison of five suspensions.

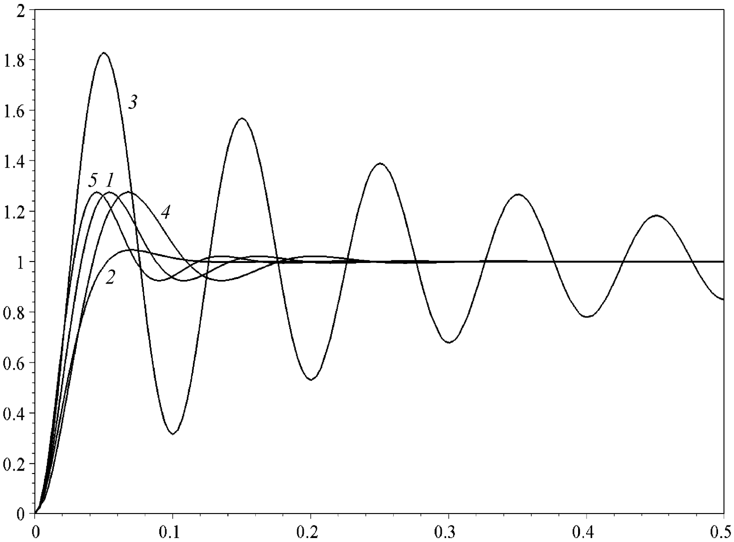

Fig. 24. Relative displacement time response of the linear mount to a unit step input, comparison of five suspensions.

7. Conclusions

Mathematically, the performance index and the definition of the cost function has a central role in the result of an optimal design of a system. Although there is no universally accepted cost function for the isolation of mechanical vibration systems even for a simple linear base excited 1-DOF vibration isolator, the main parameters included in most cost functions are known.

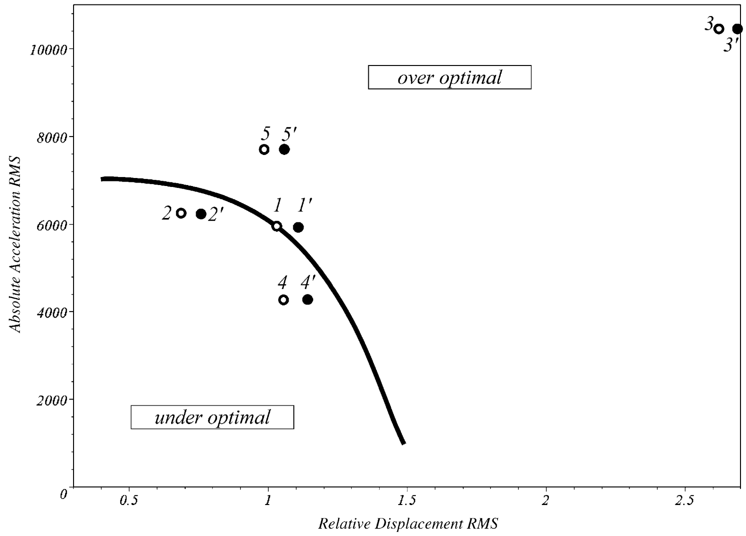

Fig. 25. Perturbed trial points for linear mount in $RMS(\ddot{x})$–$RMS(x_r)$ plane.

Table 2
Numerical values of dynamical parameters for five different perturbed suspensions shown in Fig. 25

	R	η	f_n	ξ
Point 1′	5945	1.10967	10	0.34
Point 2′	6252	0.7670	10.1	0.59
Point 3′	10447	2.7025	9.4	0.05
Point 4′	4267	1.1350	8.2	0.34
Point 5′	7701	1.0653	11.7	0.33

In this paper a novel optimization method based on the RMS of the acceleration and relative displacement is defined. Furthermore, the GA is used to apply the optimization conditions. The result demonstrates that there is an optimal relation between the natural frequency and damping ratio. The optimal natural frequency and damping ratio values of the mount, and quarter car lie on a curve connecting the minimum of the RMS absolute acceleration with respect to the RMS relative displacement. This optimum curve demonstrates that the optimal values do not lie on the boundaries of constraints.

In order to analyze the behavior of the optimal system, a comparison of a point on the line of minima with four off line suspensions is analyzed. It is shown that the frequency response of the system with optimal parameters depicted by the line of minima is better than the off optimal values.

The analysis illustrate that at optimum conditions, an increase in natural frequency should be followed with an increase in damping ratio. This phenomenon is relatively linear for the quarter car, but it is completely different for a mount suspension. In this case the rate of increasing natural frequency at low damping is much more than the rate at high frequency. Hence, adding a

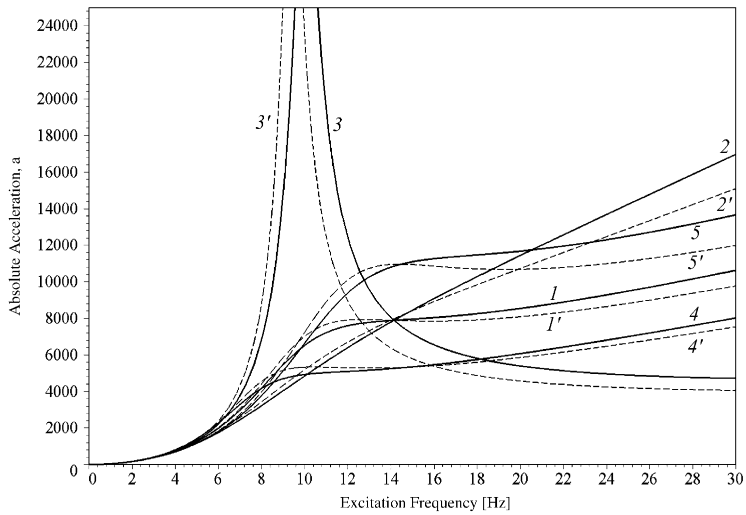

Fig. 26. Linear mount absolute acceleration frequency response a, comparison of 10 suspensions.

mechanical filter to isolate the suspended mass of a linear mount, moderate the rate of changing natural frequency and damping ratio. It should be a good idea to add another mechanical filter and apply the method to a 3-DOF to find the effect of increasing the number of filters.

Appendix A. Nomenclature

A, B, C	coefficients of a parabola
g	general function
$\mathrm{i} = \sqrt{-1}$	imaginary unit
J	cost function
t	time
$r = \omega/\omega_s$	excitation frequency ratio
$u = r^2\alpha^2\mu$	sprung mass absolute acceleration
U	sprung mass amplitude acceleration RMS
$v = r^2\alpha^2\tau$	unsprung mass absolute acceleration
X	amplitude of displacement
y	base displacement co-ordinate
Y	amplitude of base displacement excitation
$a = \lvert\ddot{X}/Y\rvert$	absolute acceleration transmissibility of mount
R	acceleration RMS of mount
η	absolute acceleration RMS of mount
e	infinitesimal increment
$\alpha = \omega_s/\omega_u$	natural frequency ratio
$\varepsilon = m_s/m_u$	mass ratio
$\mu = X_s/Y$	sprung mass transmissibility

$\eta = (X_s - X_u)/Y$	sprung mass relative displacement transmissibility
$\tau = X_u/Y$	unsprung mass transmissibility
$\xi = \dfrac{c_s}{2\omega_s m_s}$	damping ratio
Φ	relative displacement RMS
$\omega_u = \sqrt{k_u/m_u}$	unsprung mass natural frequency
$\omega_s = \sqrt{k_s/m_s}$	sprung mass natural frequency
Y	amplitude of harmonic excitation (m)
x	absolute displacement of m (m)
x_r	relative displacement of m (m)
ξ	damping ratio
ω	excitation frequency
ω_n	natural frequency in (rad/s)
f_n	natural frequency in (Hz)

Subscript

m	minimum
s	sprung
u	unsprung
i	a picked point with smaller damping than point j
j	random picked point
k	a picked point with greater damping than point j
c	critical
g	given

References

[1] J.P. Arora, *Introduction to Optimum Design*, McGraw-Hill, New York, 1989.
[2] J.H. Holland, *Adaptation in Natural and Artificial System*, University of Michigan Press, Ann Arbor, MI, 1975.
[3] M. Gen, R. Cheng, *Genetic Algorithms and Engineering Optimization*, Wiley, New York, 2000.
[4] C. Mares, C. Surace, An application of genetic algorithms to identify damage in elastic structures, *Journal of Sound and Vibration* 195 (2) (1996) 195–215.
[5] Z. Michalewicz, *Genetic Algorithms + Data Structures = Evolution Programs*, Springer, New York, 1996.
[6] K.A. DeJong, Genetic algorithms: a ten years perspective, *Proceedings of the International Conference on Genetic Algorithms and Their Applications*, Lawrence Erlbaum Associates, 1985, pp. 169–177.
[7] D.E.G. Goldberg, *Genetic Algorithms in Search, Optimization and Machine Learning*, Addison-Wesley, Reading, MA, 1989.
[8] S. Forrest, Genetic algorithms: principles of natural selection applied to computation, *Science* 261 (5123) (1993) 872–878.
[9] I.I. Esat, H. Bahai, Vibratory system synthesis for multi-body system based on genetic algorithm, *Journal of Sound and Vibration* 230 (4) (2000) 933–948.
[10] A.E. Baumal, J.J. McPhee, P.H. Calamai, Application of genetic algorithms to the design optimization of an active vehicle suspension system, *Computer Methods in Applied Mechanics and Engineering* 163 (1–4) (1998) 87–94.
[11] N. Baldanzini, D. Caprioli, M. Pierini, Designing the dynamic behavior of an engine suspension system through genetic algorithms, *Journal of Vibration and Acoustics* 123 (2001) 480–486.

[12] C.H. Hariss, *Shock and Vibration Handbook*, McGraw-Hill, New York, 1996.
[13] J.C. Snowdon, Vibration isolation: use and characterization, *Journal of the Acoustical Society of America* 66 (5) (1979) 1245–1274.
[14] F.C. Nelson, Vibration isolation: a review, I. Sinusoidal and random excitations, *Shock and Vibration* 1 (5) (1994) 485–493.
[15] J. Nakhaie, M.F. Golnaraghi, Engine mounts: a survey, *The Shock and Vibration Digest* 34 (5) (2002) 363–379.
[16] J.P. Den Hartog, *Mechanical Vibrations*, 4th Edition, McGraw-Hill, New York, 1956.
[17] A.O. Sykes, Isolation provided by vibration mounts, *SAE Transaction* 66 (1958) 532–548.
[18] B. Fazlollahi, R. Vahidov, A method for generation of alternatives by decision support system, *Journal of Management Information Systems* 18 (2) (2001) 229–250.
[19] M. Roseau, *Vibrations in Mechanical Systems*, Springer, Berlin, 1987.
[20] R.S. Sharp, D.A. Crolla, Road vehicle suspension system design: a review, *Vehicle System Dynamics* 16 (1987) 167–192.